

Nonequilibrium stationary states with Gibbs measure for two or three species of interacting particles

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Abstract. We construct explicit examples of one-dimensional driven diffusive systems for two and three species of interacting particles, defined by asymmetric dynamical rules which do not obey detailed balance, but whose nonequilibrium stationary-state measure coincides with a prescribed equilibrium Gibbs measure. For simplicity, the measures considered in this construction only involve nearest-neighbor interactions. For two species, the dynamics thus obtained generically has five free parameters, and does not obey pairwise balance in general. The latter property is satisfied only by the totally asymmetric dynamics and the partially asymmetric dynamics with uniform bias, i.e., the cases originally considered by Katz, Lebowitz, and Spohn. For three species of interacting particles, with nearest-neighbor interactions between particles of the same species, the totally asymmetric dynamics thus obtained has two free parameters, and obeys pairwise balance. These models are put in perspective with other examples of driven diffusive systems. The emerging picture is that asymmetric (nonequilibrium) stochastic dynamics leading to a given stationary-state measure are far more constrained (in terms of numbers of free parameters) than the corresponding symmetric (equilibrium) dynamics.

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1. Introduction

Driven diffusive systems [1, 2, 3, 4] are defined by stochastic dynamical rules that incorporate the effect of an external drive, and therefore do not obey detailed balance, which makes their nonequilibrium stationary states difficult to study in general. The first step of evaluating the stationary-state measure, i.e., the probability $P_{\text{st}}(\mathcal{C})$ of any given configuration \mathcal{C} , is already a non-trivial task. In view of the lack of a general theory of nonequilibrium stationary states, one has to rely on the investigation of specific models, for which the stationary-state measure is analytically tractable. Examples of such models are the simple exclusion processes, the zero-range process, and the one-dimensional Katz-Lebowitz-Spohn (KLS) model.

For the simple exclusion process, particles obeying an exclusion constraint perform random walks on a lattice, with either symmetric or biased moves [3, 4, 5]. On a ring, i.e., with periodic boundary conditions, the stationary-state measure is uniform, irrespectively of the bias: all the configurations are equally probable. The same model with open boundaries has a stationary-state measure which can be described in terms of a matrix product Ansatz. The zero-range process (ZRP) [6], in the presence of a bias, belongs to the class of driven stochastic processes with multiple occupancies. Its stationary state has a product measure, again irrespectively of the bias. The occupation numbers of the sites are independent random quantities with a common distribution, up to the conservation of the total number of particles. The existence of unbounded occupancies however opens up the possibility of having a condensation transition, irrespectively of the geometry of the underlying lattice, and therefore also in one dimension [7, 8].

The KLS model is a lattice gas model of interacting charged particles subjected to an external electric field [9]. It is representative of the class of models with non-equilibrium stationary state measures incorporating physical interactions between particles. The stationary-state measure of the KLS model is non trivial in two dimensions and above, where e.g. the critical temperature depends continuously on the applied field [2]. The situation however simplifies in the one-dimensional geometry, where the model is equivalent to a chain of classical Ising spins $s_n = \pm 1$. In this case, there exists a class of biased stochastic dynamics, for which the nonequilibrium stationary-state measure is the canonical finite-temperature Gibbs measure, where the probability $P_{\text{st}}(\mathcal{C})$ of the configuration $\mathcal{C} = \{s_1, \dots, s_N\}$ is given by the Boltzmann formula (with $k_B T = 1$)

$$P_{\text{st}}(\mathcal{C}) = \frac{1}{Z} \exp(-\mathcal{H}(\mathcal{C})) \quad (1.1)$$

associated with the Ising Hamiltonian with nearest-neighbor interactions

$$\mathcal{H} = -J \sum_n s_n s_{n+1}. \quad (1.2)$$

This very stochastic model with antiferromagnetic interactions ($J < 0$) was subsequently rediscovered in the context of polymer crystallization [10].

At this point it is natural to question the generality of the result found in [9] for the case of the Ising chain. In the present work we show that for systems with three species

of particles there also exist asymmetric stochastic dynamics which do not obey detailed balance, but whose stationary-state measure is the Gibbs measure corresponding to a simple Hamiltonian. In particular this stationary-state measure is independent of the bias. In our construction we restrict the choice of measures to those involving only nearest-neighbor interactions. We first revisit in Section 2 the case of two species of interacting particles, thus generalizing the study of [9] to a wider class of dynamics. The emphasis is put on the role of various symmetries, and especially on the number of free parameters left by imposing them. We then consider, in Section 3, the entirely novel situation of three species of interacting particles. Section 4 contains a discussion, where our results are put in perspective with yet other examples.

Let us finally give a brief reminder of the concepts of detailed balance [11] and pairwise balance [12], which will be used throughout this work. Consider a finite set of configurations \mathcal{C} , and a Markovian dynamics in continuous time, defined by the transition rates $W(\mathcal{C} \rightarrow \mathcal{C}')$. The master equation for the time-dependent probability $P(\mathcal{C}, t)$ reads

$$\frac{dP(\mathcal{C}, t)}{dt} = \sum_{\mathcal{C}'} (W(\mathcal{C}' \rightarrow \mathcal{C})P(\mathcal{C}', t) - W(\mathcal{C} \rightarrow \mathcal{C}')P(\mathcal{C}, t)). \quad (1.3)$$

The stationary probability $P_{\text{st}}(\mathcal{C})$ therefore obeys the equation

$$\sum_{\mathcal{C}'} (W(\mathcal{C}' \rightarrow \mathcal{C})P_{\text{st}}(\mathcal{C}') - W(\mathcal{C} \rightarrow \mathcal{C}')P_{\text{st}}(\mathcal{C})) = 0. \quad (1.4)$$

In the particular case where the Markov process is reversible, the dynamics brings the system to an equilibrium state. Reversibility requires the *detailed balance* property [11], that is the absence of probability flux between any pair of configurations \mathcal{C} and \mathcal{C}' at stationarity:

$$W(\mathcal{C} \rightarrow \mathcal{C}')P_{\text{st}}(\mathcal{C}) = W(\mathcal{C}' \rightarrow \mathcal{C})P_{\text{st}}(\mathcal{C}'). \quad (1.5)$$

Equation (1.5) clearly implies (1.4).

A weaker property, referred to as *pairwise balance* [12], is adapted to the situation of driven diffusive systems, where the presence of a preferred direction of motion, i.e., a bias, precludes the property of detailed balance. Pairwise balance is defined as follows: for every pair of configurations \mathcal{C} and \mathcal{C}' such that $W(\mathcal{C} \rightarrow \mathcal{C}') \neq 0$, there exists a third configuration \mathcal{C}'' such that

$$W(\mathcal{C} \rightarrow \mathcal{C}')P_{\text{st}}(\mathcal{C}) = W(\mathcal{C}'' \rightarrow \mathcal{C})P_{\text{st}}(\mathcal{C}''). \quad (1.6)$$

The moves $\mathcal{C} \rightarrow \mathcal{C}'$ and $\mathcal{C}'' \rightarrow \mathcal{C}$ are said to be conjugate to each other. Equation (1.6) also implies (1.4), since terms corresponding to pairs of conjugate moves cancel each other. Several of the examples of dynamics constructed in this paper obey P-related pairwise balance, where pairs of conjugate moves are related to each other by parity. The first explicit example is given below (2.20). An important general consequence of P-related pairwise balance is shown at the end of Section 3. If the totally asymmetric dynamics obeys P-related pairwise balance, the partially asymmetric dynamics with uniform bias p also obeys pairwise balance, and it has the same stationary-state measure as the totally asymmetric one, irrespective of the value of the bias.

2. Two species

Consider a ring made of N sites. Each site is occupied by a particle, which can be either of type A (positively charged) or of type B (negatively charged). We represent the species of particle at site n by an Ising spin $s_n = \pm 1$ equal to the charge of the particle. Table 1 also gives the corresponding indicator variables. For instance, $(1 + s_n)/2$ is equal to 1 if the particle at site n is of type A , and to 0 else.

Table 1. Spin (charge) and indicator variables associated with each particle species in the case of two species.

Species at site n	Spin (charge)	Indicator variable
A	$s_n = +1$	$(1 + s_n)/2$
B	$s_n = -1$	$(1 - s_n)/2$

Our goal is to construct nonequilibrium dynamics such that the stationary-state measure is the Gibbs measure given by (1.1) associated with the nearest-neighbor Hamiltonian (1.2). We consider the asymmetric exchange (Kawasaki) dynamics. Consistently with the form of the Hamiltonian (1.2), the rates depend on the two neighbors of the pair to be exchanged, according to Table 2. The dynamics so defined conserves the numbers N_A and N_B of particles of each species, with $N_A + N_B = N$. We first determine the most general dynamics of this form in Section 2.1, and then discuss the interplay between various possible symmetries in Section 2.2.

Table 2. List of moves in the general exchange dynamics for two species of interacting particles, notation for the corresponding exchange rates, and energy difference $\Delta\mathcal{H}$ involved in the moves, where the Hamiltonian \mathcal{H} is defined in (1.2).

Move	Rate	$\Delta\mathcal{H}$	Move	Rate	$\Delta\mathcal{H}$
$AABA \rightarrow ABAA$	w_{AA}	0	$ABAA \rightarrow AABA$	x_{AA}	0
$AABB \rightarrow ABAB$	w_{AB}	$4J$	$ABAB \rightarrow AABB$	x_{AB}	$-4J$
$BABA \rightarrow BBAA$	w_{BA}	$-4J$	$BBAA \rightarrow BABA$	x_{BA}	$4J$
$BABB \rightarrow BBAB$	w_{BB}	0	$BBAB \rightarrow BABB$	x_{BB}	0

2.1. The general case

Consider the numbers N_{AA}, \dots, N_{BB} of oriented pairs of neighbors of each species. These numbers obey the sum rules

$$N_A = N_{AA} + N_{AB} = N_{AA} + N_{BA}, \quad N_B = N_{BA} + N_{BB} = N_{AB} + N_{BB}. \quad (2.1)$$

The sum of the two equations gives twice the same equation, so that the four pair numbers obey three independent equations, leaving one single free quantity. It is

convenient to take the latter as being the Hamiltonian \mathcal{H} of (1.2). The pair numbers can indeed be expressed as linear combinations of \mathcal{H} and of the particle numbers N_A and N_B :

$$\begin{aligned} N_{AA} &= \frac{1}{4} (3N_A - N_B - \mathcal{H}/J), & N_{BB} &= \frac{1}{4} (3N_B - N_A - \mathcal{H}/J), \\ N_{AB} &= N_{BA} = \frac{1}{4} (N + \mathcal{H}/J). \end{aligned} \quad (2.2)$$

Incidentally, this proves that the Hamiltonian (1.2) is the most general form of a pair Hamiltonian for two species of particles with nearest-neighbor interactions.

Throughout this paper, we make use of an alternative and convenient way of automatically taking into account sum rules such as (2.1). This consists in expressing all the quantities in terms of the spin variables s_n introduced in Table 1. For instance

$$N_A = \frac{1}{2} \sum_n (1 + s_n) = \frac{N}{2} (1 + \langle s_1 \rangle). \quad (2.3)$$

Here and in the following, the brackets $\langle \dots \rangle$ denote a uniform spatial average *for a fixed generic configuration* \mathcal{C} . Recall that all the sites are equivalent, because of translational invariance. The pair numbers and the Hamiltonian read

$$\begin{aligned} N_{AA} &= \frac{N}{4} (1 + 2\langle s_1 \rangle + \langle s_1 s_2 \rangle), & N_{BB} &= \frac{N}{4} (1 - 2\langle s_1 \rangle + \langle s_1 s_2 \rangle), \\ N_{AB} &= N_{BA} = \frac{N}{4} (1 - \langle s_1 s_2 \rangle), & \mathcal{H} &= -NJ \langle s_1 s_2 \rangle. \end{aligned} \quad (2.4)$$

Equations such as (2.1) and (2.2) are then automatically satisfied.

Consider now the fate of a generic configuration \mathcal{C} . The total exit rate $W_{\text{out}}(\mathcal{C})$ from \mathcal{C} to any other configuration \mathcal{C}' per unit time can be read off from Table 2:

$$\begin{aligned} W_{\text{out}}(\mathcal{C}) &= w_{AA}N_{AABA} + w_{AB}N_{AABB} + w_{BA}N_{BABA} + w_{BB}N_{BABB} \\ &\quad + x_{AA}N_{ABAA} + x_{AB}N_{ABAB} + x_{BA}N_{BBAA} + x_{BB}N_{BBAB}. \end{aligned} \quad (2.5)$$

An analogous expression can be derived for the total entrance rate $W_{\text{in}}(\mathcal{C})$ from any other configuration \mathcal{C}' to \mathcal{C} . Using again Table 2, as well as (1.1) to express the stationary-state weight $P_{\text{st}}(\mathcal{C}')$ as

$$P_{\text{st}}(\mathcal{C}') = P_{\text{st}}(\mathcal{C}) \exp(\Delta\mathcal{H}), \quad (2.6)$$

in terms of $P_{\text{st}}(\mathcal{C})$ and of the energy difference

$$\Delta\mathcal{H} = \mathcal{H}(\mathcal{C}) - \mathcal{H}(\mathcal{C}') \quad (2.7)$$

involved in the move, we obtain

$$\begin{aligned} W_{\text{in}}(\mathcal{C}) &= w_{AA}N_{ABAA} + e^{4J}w_{AB}N_{ABAB} + e^{-4J}w_{BA}N_{BBAA} + w_{BB}N_{BBAB} \\ &\quad + x_{AA}N_{AABA} + e^{-4J}x_{AB}N_{AABB} + e^{4J}x_{BA}N_{BABA} + x_{BB}N_{BABB}. \end{aligned} \quad (2.8)$$

In the stationary state we have

$$W_{\text{out}}(\mathcal{C}) - W_{\text{in}}(\mathcal{C}) = 0 \quad (2.9)$$

for every configuration \mathcal{C} . In order to determine the number of independent conditions on the rates imposed by this equation, it is convenient to rewrite (2.5) and (2.8) in terms of spin correlations, i.e., spatial averages of products of spin variables, denoted as $\langle \dots \rangle$, along the lines of (2.3). With these notations, we obtain

$$W_{\text{out}}(\mathcal{C}) - W_{\text{in}}(\mathcal{C}) = \frac{N}{16} \left\{ (e^{-4J} - 1)(\langle s_1 s_2 s_3 s_4 \rangle + 1) R_1 + \langle s_1(s_3 - s_2)s_4 \rangle R_2 \right. \\ \left. + \left[\langle s_1(3s_2 - 2s_3 + s_4) \rangle + e^{-4J} \langle s_1(s_2 - 2s_3 - s_4) \rangle \right] R_1 \right\}, \quad (2.10)$$

where R_1 and R_2 stand for the following linear combinations of the rates:

$$R_1 = e^{4J}(w_{AB} + x_{BA}) - w_{BA} - x_{AB}, \\ R_2 = (e^{4J} + 1)(w_{AB} - x_{BA}) + (e^{-4J} + 1)(w_{BA} - x_{AB}) \\ - 2(w_{AA} + w_{BB} - x_{AA} - x_{BB}). \quad (2.11)$$

The condition (2.9) therefore gives two linear relations,

$$R_1 = R_2 = 0, \quad (2.12)$$

between the eight exchange rates defining the general asymmetric dynamics. Let us choose the time unit by setting

$$w_{AA} + w_{BB} + x_{AA} + x_{BB} = 1. \quad (2.13)$$

The most general asymmetric dynamics for two species of interacting particles such that the stationary-state measure is given by (1.1), (1.2) therefore has five free parameters. An explicit parametrization of the rates is given in Table 3. The dynamics thus obtained does not obey pairwise balance in general.

The parametrization of the solutions to (2.12) and (2.13) given in Table 3 has been carefully chosen in such a way that the various symmetries to be described below correspond to the simple constraints (2.17), (2.21), (2.25), (2.27) in terms of the parameters δ and $\varepsilon_1, \dots, \varepsilon_4$. The parameters δ and $\varepsilon_1, \dots, \varepsilon_4$ all lie in the range $[-1, +1]$, and are such that the combination

$$\lambda = \frac{(1 + \delta)\varepsilon_1 + (1 - \delta)\varepsilon_2}{\varepsilon_3 + \varepsilon_4} \quad (2.14)$$

is positive.

2.2. The interplay between various symmetries

The number of free parameters of the dynamics thus obtained is decreased if various kinds of symmetries are imposed onto the dynamics.

- *Symmetric (P-invariant) dynamics.* Consider a symmetric dynamics, invariant under the spatial parity P which reverses the orientation of the ring (i.e., interchanges left and right). This symmetry property reads

$$x_{IJ} = w_{JI} \quad (2.15)$$

Table 3. Explicit parametrization of the rates of the most general asymmetric dynamics for two species with Gibbs stationary-state measure (1.1), (1.2). The notation λ is defined in (2.14).

Rate	expression	Rate	expression
w_{AA}	$\frac{(1 + \varepsilon_1)(1 + \delta)}{4}$	x_{AA}	$\frac{(1 - \varepsilon_1)(1 + \delta)}{4}$
w_{AB}	$\frac{(1 + \varepsilon_3)\lambda}{2(e^{4J} + 1)}$	x_{AB}	$\frac{(1 - \varepsilon_4)\lambda e^{4J}}{2(e^{4J} + 1)}$
w_{BA}	$\frac{(1 + \varepsilon_4)\lambda e^{4J}}{2(e^{4J} + 1)}$	x_{BA}	$\frac{(1 - \varepsilon_3)\lambda}{2(e^{4J} + 1)}$
w_{BB}	$\frac{(1 + \varepsilon_2)(1 - \delta)}{4}$	x_{BB}	$\frac{(1 - \varepsilon_2)(1 - \delta)}{4}$

for all values of the indices $I, J = A, B$. The stationary state thus obtained is an equilibrium state. The first equation of (2.12),

$$w_{BA} = e^{4J} w_{AB}, \quad (2.16)$$

expresses detailed balance. Equation (2.15) amounts to setting

$$\varepsilon_i = 0 \quad (i = 1, \dots, 4) \quad (2.17)$$

in Table 3. The symmetric (equilibrium) dynamics therefore has two free parameters: δ and λ , in the ranges $-1 < \delta < 1$ and $\lambda > 0$. The expression (2.14) for λ indeed becomes indeterminate in the limit where all the ε_i go simultaneously to zero. The rates read

$$\begin{aligned} w_{AA} = x_{AA} &= \frac{1 + \delta}{4}, & w_{AB} = x_{BA} &= \frac{\lambda}{2(e^{4J} + 1)}, \\ w_{BA} = x_{AB} &= \frac{\lambda e^{4J}}{2(e^{4J} + 1)}, & w_{BB} = x_{BB} &= \frac{1 - \delta}{4}. \end{aligned} \quad (2.18)$$

• *Totally asymmetric dynamics.* Consider a dynamics driven by an infinitely strong electric field, so that the positively (resp. negatively) charged A particles (resp. B particles) hop exclusively to the right (resp. to the left). Therefore

$$x_{IJ} = 0 \quad (2.19)$$

for all values of the indices $I, J = A, B$. Equation (2.12) becomes

$$w_{BA} = e^{4J} w_{AB}. \quad (2.20)$$

This equation coincides with (2.16). It expresses *P-related pairwise balance*: conjugate moves are related to each other by parity P , i.e., the first and the fourth move of the left column of Table 2 are their own conjugates, whereas the second and the third moves are conjugate to each other. Equation (2.19) amounts to setting

$$\varepsilon_i = 1 \quad (i = 1, \dots, 4) \quad (2.21)$$

in Table 3, so that $\lambda = 1$. The totally asymmetric dynamics with stationary-state measure (1.1), (1.2) therefore has one free parameter: δ , in the range $-1 < \delta < 1$. The rates read

$$\begin{aligned} w_{AA} &= \frac{1+\delta}{2}, & w_{AB} &= \frac{1}{e^{4J}+1}, \\ w_{BA} &= \frac{e^{4J}}{e^{4J}+1}, & w_{BB} &= \frac{1-\delta}{2}. \end{aligned} \quad (2.22)$$

- *Partially asymmetric dynamics with a uniform bias.* This is the most general case originally considered by KLS [9]. Consider a dynamics driven by a finite electric field, so that the positively (resp. negatively) charged A particles (resp. B particles) hop preferentially to the right (resp. to the left). Let

$$p = \frac{1+\varepsilon}{2}, \quad q = \frac{1-\varepsilon}{2} \quad (2.23)$$

be the a priori probabilities of respectively hopping to the right and to the left, where $0 \leq \varepsilon \leq 1$ provides a measure of the applied electric field. This translates into the following uniform bias condition:

$$\frac{x_{IJ}}{w_{JI}} = \frac{1-\varepsilon}{1+\varepsilon} \quad (2.24)$$

for all values of $I, J = A, B$. This situation interpolates between the symmetric case ($p = 1/2$, $\varepsilon = 0$) and the totally asymmetric one ($p = 1$, $\varepsilon = 1$). Equation (2.24) amounts to setting

$$\varepsilon_i = \varepsilon \quad (i = 1, \dots, 4) \quad (2.25)$$

in Table 3, so that again $\lambda = 1$. As a consequence, there is a two-parameter family of dynamics with uniform bias and stationary-state measure (1.1), (1.2), parametrized by ε and δ . We thus recover the original KLS model [9]. The fact that the stationary-state weights are independent of the bias is actually a general property of dynamics obeying P-related pairwise balance (see Section 3.2).

- *CP-invariance.* The CP operation is the product of C and P, where the charge conjugation C changes the charge of the particles to its opposite (i.e., interchanges A and B particles), whereas the spatial parity P changes the orientation of the ring (i.e., interchanges left and right). In physical terms, in the stationary state of a CP-invariant dynamics, the current due to a positively charged particle and to a negatively charged particle are equal. Requiring CP-invariance yields the two conditions

$$w_{AA} = w_{BB}, \quad x_{AA} = x_{BB}, \quad (2.26)$$

which amount to setting

$$\delta = 0, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon \quad (2.27)$$

in Table 3. The most general CP-invariant dynamics has therefore three free parameters: ε , ε_3 , and ε_4 . It does not obey pairwise balance in general.

CP-invariance can be combined with any of the above symmetries:

★ The CP-invariant symmetric dynamics corresponds to $\delta = \varepsilon_i = 0$. It has a single free parameter: λ . The rates read

$$\begin{aligned} w_{AA} = x_{AA} &= \frac{1}{4}, & w_{AB} = x_{BA} &= \frac{\lambda}{2(e^{4J} + 1)}, \\ w_{BA} = x_{AB} &= \frac{\lambda e^{4J}}{2(e^{4J} + 1)}, & w_{BB} = x_{BB} &= \frac{1}{4}. \end{aligned} \quad (2.28)$$

★ The partially asymmetric CP-invariant dynamics with uniform bias has one single free parameter: ε .

★ The totally asymmetric CP-invariant dynamics is the most constrained of all the dynamics: it has no free parameter at all. The rates

$$w_{AA} = w_{BB} = \frac{1}{2}, \quad w_{AB} = \frac{1}{e^{4J} + 1}, \quad w_{BA} = \frac{e^{4J}}{e^{4J} + 1} \quad (2.29)$$

only depend on the energy difference $\Delta\mathcal{H}$ involved in the exchange moves. They coincide with those of the heat-bath rule [13, 14]:

$$w(\Delta\mathcal{H}) = \frac{1}{\exp(\Delta\mathcal{H}) + 1} = \frac{1}{2} \left(1 - \tanh \frac{\Delta\mathcal{H}}{2} \right). \quad (2.30)$$

The above discussion is summarized in Table 4, giving the number of free parameters for every symmetry class of dynamics, both without and with imposing CP-invariance.

Table 4. List of the symmetry classes of dynamics for two species of particles with Gibbs stationary-state measure (1.1), (1.2), with balance property: detailed balance (DB) or pairwise balance (PB), and number of free parameters, both without and with CP-invariance.

Class of dynamics	Balance property	Without CP	With CP
General	none	5	3
Symmetric (equilibrium)	DB	2	1
Totally asymmetric	PB	1	0
Partially asymmetric (uniform bias)	PB	2	1

3. Three species

Consider again a finite ring of N sites. Each site is now occupied by a particle which can be either of type A (positively charged), of type B (negatively charged), or of type C (neutral, i.e., with no charge). We again represent the species of particle at site n by a spin $S_n = 0, \pm 1$ equal to the charge of the particle, as shown in Table 5.

We consider Gibbs measures corresponding to the most general (ferromagnetic or antiferromagnetic) Hamiltonian involving pairs of identical nearest neighbors:

$$\mathcal{H} = -2(J_A N_{AA} + J_B N_{BB} + J_C N_{CC}), \quad (3.1)$$

Table 5. Spin (charge) and indicator variables associated with each particle species in the case of three species.

Species at site n	Spin (charge)	Indicator variable
A	$S_n = +1$	$S_n(S_n + 1)/2$
B	$S_n = -1$	$S_n(S_n - 1)/2$
C	$S_n = 0$	$1 - S_n^2$

where the coupling constants J_A , J_B , and J_C can take both signs. The factor 2 is introduced for consistency with the case of two species. Using the spin variables $S_n = 0, \pm 1$ defined in Table 5, the Hamiltonian (3.1) can be rewritten as

$$\mathcal{H} = E_0 - \frac{1}{2} \sum_n \left[(J_A + J_B + 4J_C) S_n S_{n+1} + (J_A - J_B)(S_n + S_{n+1}) + J_A + J_B \right] S_n S_{n+1}, \quad (3.2)$$

where $E_0 = 2(N - 2N_C)J_C$ is a constant. This is a generalized Blume-Emery-Griffiths spin-1 Hamiltonian [15].

We again address the question of the existence of nonequilibrium stochastic dynamics whose stationary-state measure is the measure (1.1) associated with the Hamiltonian (3.1). The results obtained in Section 2 for two species of particles suggest that the case of totally asymmetric exchange dynamics is already of interest. We therefore restrict our investigation to this limiting situation for the time being. The positively (resp. negatively) charged A particles (resp. B particles) only hop to the right (resp. to the left), whereas the neutral C particles can hop in both directions. The exchange rates depend on the two neighbors of the pair to be exchanged, according to Table 6. The dynamics so defined conserves the numbers N_A , N_B , and N_C of particles of each species, with $N_A + N_B + N_C = N$.

3.1. The CP-invariant case

Motivated by the form of the results of Section 2 on two species, we first consider the CP-invariant case, which can be anticipated to be simpler than the generic one.

As far as statics is concerned, C-invariance implies

$$J_A = J_B = J, \quad J_C = J_0. \quad (3.3)$$

The Hamiltonian (3.2) becomes the usual Blume-Emery-Griffiths Hamiltonian [15]

$$\mathcal{H} = E_0 - \sum_n \left[(J + 2J_0) S_n S_{n+1} + J \right] S_n S_{n+1}. \quad (3.4)$$

As far as dynamics is concerned, CP-invariance yields 12 equalities among the 27 exchange rates:

$$\begin{aligned} w_{AA} &= w_{BB}, & w_{AC} &= w_{CB}, & w_{BC} &= w_{CA}, & x_{AA} &= y_{BB}, \\ x_{AB} &= y_{AB}, & x_{AC} &= y_{CB}, & x_{BA} &= y_{BA}, & x_{BB} &= y_{AA}, \\ x_{BC} &= y_{CA}, & x_{CA} &= y_{BC}, & x_{CB} &= y_{AC}, & x_{CC} &= y_{CC}. \end{aligned} \quad (3.5)$$

Table 6. List of moves in the totally asymmetric dynamics for three species of interacting particles, notation for the corresponding exchange rates, and energy difference $\Delta\mathcal{H}$ involved in the moves, where the Hamiltonian \mathcal{H} is defined in (3.1).

Move	Rate	$\Delta\mathcal{H}$	Move	Rate	$\Delta\mathcal{H}$
$AABA \rightarrow ABAA$	w_{AA}	0	$BCBC \rightarrow BBCC$	x_{BC}	$-2J_B - 2J_C$
$AABB \rightarrow ABAB$	w_{AB}	$2J_A + 2J_B$	$CCBA \rightarrow CBCA$	x_{CA}	$2J_C$
$AABC \rightarrow ABAC$	w_{AC}	$2J_A$	$CCBB \rightarrow CBCB$	x_{CB}	$2J_B + 2J_C$
$BABA \rightarrow BBAA$	w_{BA}	$-2J_A - 2J_B$	$CCBC \rightarrow CBCC$	x_{CC}	0
$BABB \rightarrow BBAB$	w_{BB}	0	$AACA \rightarrow ACAA$	y_{AA}	0
$BABC \rightarrow BBAC$	w_{BC}	$-2J_B$	$AACB \rightarrow ACAB$	y_{AB}	$2J_A$
$CABA \rightarrow CBAA$	w_{CA}	$-2J_A$	$AACC \rightarrow ACAC$	y_{AC}	$2J_A + 2J_C$
$CABB \rightarrow CBAB$	w_{CB}	$2J_B$	$BACA \rightarrow BCAA$	y_{BA}	$-2J_A$
$CABC \rightarrow CBAC$	w_{CC}	0	$BACB \rightarrow BCAB$	y_{BB}	0
$ACBA \rightarrow ABCA$	x_{AA}	0	$BACC \rightarrow BCAC$	y_{BC}	$2J_C$
$ACBB \rightarrow ABCB$	x_{AB}	$2J_B$	$CACA \rightarrow CCAA$	y_{CA}	$-2J_A - 2J_C$
$ACBC \rightarrow ABCC$	x_{AC}	$-2J_C$	$CACB \rightarrow CCAB$	y_{CB}	$-2J_C$
$BCBA \rightarrow BBCA$	x_{BA}	$-2J_B$	$CACC \rightarrow CCAC$	y_{CC}	0
$BCBB \rightarrow BB CB$	x_{BB}	0			

The analysis follows the lines of Section 2. The algebra is however far more cumbersome, so that intermediate expressions are too lengthy to be reported here. Calculations have been worked out with the help of the software MACSYMA. We start from the expressions for the total rates $W_{\text{out}}(\mathcal{C})$ and $W_{\text{in}}(\mathcal{C})$ for a generic configuration \mathcal{C} , similar to (2.5) and (2.8), which can be read off from Table 6. The difference $W_{\text{out}}(\mathcal{C}) - W_{\text{in}}(\mathcal{C})$ is then recast in terms of products of the spin variables S_n . We thus obtain an expression similar to (2.10), involving 42 different correlations of two to eight spin variables. One example of a correlation of two variables is $\langle S_1 S_2 \rangle$, whereas there is a unique correlation of eight variables: $\langle S_1^2 S_2^2 S_3^2 S_4^2 \rangle$. Requiring that the coefficients of all these correlations vanish, we thus obtain 42 linear (but not independent) relations of the form

$$R_1 = \dots = R_{42} = 0, \quad (3.6)$$

where the R_i are linear combinations of the exchange rates, similar to R_1 and R_2 given in (2.11). In the CP-invariant situation under study, equations (3.5) and (3.6) together yield 26 independent linear relations among the 27 exchange rates, and therefore leave a single free parameter, which can be fixed by choosing the time unit. For consistency with the case of two species, we set

$$w_{AA} + w_{BB} = 1. \quad (3.7)$$

This normalization condition uniquely determines all the exchange rates.

We have therefore shown that there is a single totally asymmetric CP-invariant dynamics for three species of interacting particles with stationary-state meas-

ure (1.1), (3.1). This uniquely determined dynamics can be viewed as a non-trivial extension to three species of the result (2.29). The explicit expressions of the exchange rates are given in Table 7. This dynamics obeys pairwise balance (see equation (3.11) below for the general case). However, at variance with the case of two species, the rates are not of the heat-bath form (2.30).

Table 7. Expressions of the rates of the totally asymmetric CP-invariant dynamics for three species. The label I stands for any particle species ($I = A, B, C$).

Rate	expression	Rate	expression	Rate	expression
$w_{AA} = w_{BB}$	$\frac{1}{2}$	w_{BA}	$\frac{e^{4J}}{e^{4J} + 1}$	$x_{IA} = y_{BI}$	$\frac{e^{2J}}{2(e^{4J} + 1)}$
w_{AB}	$\frac{1}{e^{4J} + 1}$	$w_{BC} = w_{CA}$	$\frac{e^{2J}(e^{2J} + 1)}{2(e^{4J} + 1)}$	$x_{IB} = y_{AI}$	$\frac{1}{2(e^{4J} + 1)}$
$w_{AC} = w_{CB}$	$\frac{e^{2J} + 1}{2(e^{4J} + 1)}$	w_{CC}	$\frac{e^{2J}}{e^{4J} + 1}$	$x_{IC} = y_{CI}$	$\frac{e^{2J+2J_0}}{2(e^{4J} + 1)}$

3.2. The general case

We now turn to the general totally asymmetric dynamics. We view J_A , J_B , and J_C as three independent coupling constants, and consider the 27 exchange rates entering Table 6 as being a priori all different from each other.

Following the above procedure, and choosing time units according to (3.7), we are left after some lengthy algebra with a two-parameter family of dynamics with stationary-state measure (1.1), (3.1). An explicit parametrization of the rates is given in Table 8, where the parameters α and β enter linearly, and with the notation

$$f = \frac{1}{2(e^{2J_A+2J_B} + 1)}. \quad (3.8)$$

The form of the CP-invariant rates given in Table 7 has been helpful in working out this parametrization of the general case.

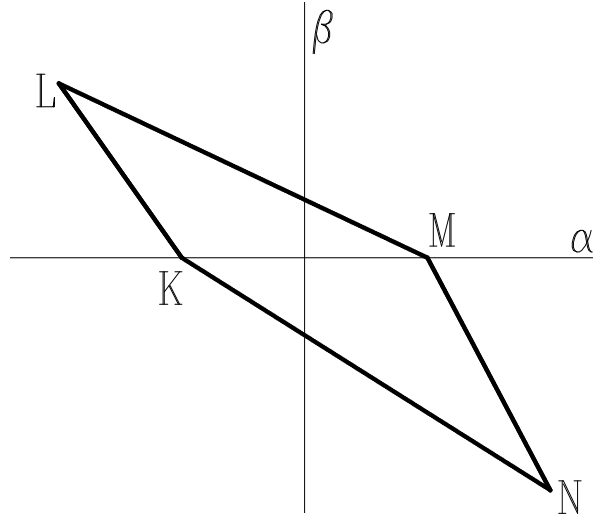
The two parameters α and β run over some domain D , such that all the rates of Table 8 are positive. It can be checked that D is an asymmetric quadrilateral, shown schematically in Figure 1. The co-ordinates of its vertices read

$$\begin{aligned} \alpha_K = -1, \quad \alpha_L = -\frac{L_A + S_B}{L_A - S_B}, \quad \alpha_M = 1, \quad \alpha_N = \frac{L_B + S_A}{L_B - S_A}, \\ \beta_K = 0, \quad \beta_L = \frac{2}{L_A - S_B}, \quad \beta_M = 0, \quad \beta_N = -\frac{2}{L_B - S_A}, \end{aligned} \quad (3.9)$$

where L_A (resp. S_A) is the largest (resp. the smallest) of the three quantities $\exp(2J_A + 2J_C)$, $\exp(-2J_A + 2J_C)$, and $\exp(-2J_A - 2J_C)$, and L_B (resp. S_B) is the largest (resp. the smallest) of the three quantities $\exp(2J_B + 2J_C)$, $\exp(-2J_B + 2J_C)$, and $\exp(-2J_B - 2J_C)$.

Table 8. Explicit parametrization of rates of the most general totally asymmetric dynamics for three species. The notation f is defined in (3.8).

Rate	expression	Rate	expression
w_{AA}	$(1 - \alpha + (1 + \alpha)e^{2J_A+2J_B})f$	x_{BC}	$(1 + \alpha + \beta)e^{2J_B+2J_C}f$
w_{AB}	$2f$	x_{CA}	$((1 + \alpha)e^{2J_B} + \beta)f$
w_{AC}	$(1 - \alpha + (1 + \alpha)e^{2J_B})f$	x_{CB}	$(1 + \alpha + \beta)f$
w_{BA}	$2e^{2J_A+2J_B}f$	x_{CC}	$((1 + \alpha)e^{2J_B+2J_C} + \beta)f$
w_{BB}	$(1 + \alpha + (1 - \alpha)e^{2J_A+2J_B})f$	y_{AA}	$(1 - \alpha - \beta e^{2J_A+2J_C})f$
w_{BC}	$(1 + \alpha + (1 - \alpha)e^{2J_A})e^{2J_B}f$	y_{AB}	$(1 - \alpha - \beta e^{2J_C})f$
w_{CA}	$(1 - \alpha + (1 + \alpha)e^{2J_B})e^{2J_A}f$	y_{AC}	$(1 - \alpha - \beta)f$
w_{CB}	$(1 + \alpha + (1 - \alpha)e^{2J_A})f$	y_{BA}	$(1 - \alpha - \beta e^{2J_C})e^{2J_A}f$
w_{CC}	$((1 - \alpha)e^{2J_A} + (1 + \alpha)e^{2J_B})f$	y_{BB}	$((1 - \alpha)e^{2J_A} - \beta e^{2J_C})f$
x_{AA}	$((1 + \alpha)e^{2J_B} + \beta e^{2J_C})f$	y_{BC}	$((1 - \alpha)e^{2J_A} - \beta)f$
x_{AB}	$(1 + \alpha + \beta e^{2J_C})f$	y_{CA}	$(1 - \alpha - \beta)e^{2J_A+2J_C}f$
x_{AC}	$((1 + \alpha)e^{2J_B} + \beta)e^{2J_C}f$	y_{CB}	$((1 - \alpha)e^{2J_A} - \beta)e^{2J_C}f$
x_{BA}	$(1 + \alpha + \beta e^{2J_C})e^{2J_B}f$	y_{CC}	$((1 - \alpha)e^{2J_A+2J_C} - \beta)f$
x_{BB}	$(1 + \alpha + \beta e^{2J_B+2J_C})f$		

**Figure 1.** Typical shape of the quadrilateral domain D of the α - β plane such that all the rates of Table 8 are positive (see text).

The general dynamics of Table 8 contains as special cases several of the situations considered so far. The CP-invariant situation corresponds to $\alpha = \beta = 0$. The rates of Table 7 are thus recovered, with the notations (3.3). The case of two species is also recovered. The rates (2.22) are reproduced, again with the notations (3.3), and with

the identification

$$\delta = \frac{e^{4J} - 1}{e^{4J} + 1} \alpha. \quad (3.10)$$

The family of dynamics described in Table 8 obeys P-related pairwise balance. This property is expressed by the nine relations

$$\begin{aligned} w_{BA} &= e^{2J_A+2J_B} w_{AB}, & w_{CA} &= e^{2J_A} w_{AC}, & w_{BC} &= e^{2J_B} w_{CB}, \\ x_{BA} &= e^{2J_B} x_{AB}, & x_{AC} &= e^{2J_C} x_{CA}, & x_{BC} &= e^{2J_B+2J_C} x_{CB}, \\ y_{BA} &= e^{2J_A} y_{AB}, & y_{CA} &= e^{2J_A+2J_C} y_{AC}, & y_{CB} &= e^{2J_C} y_{BC}, \end{aligned} \quad (3.11)$$

which are identically fulfilled by the rates of Table 8, for any values of the parameters α and β . In other words, the relations (3.11) are built in as a subset of (3.6).

Pairwise balance has the following consequence. Consider the partially asymmetric dynamics with uniform bias p , where all the ‘right’ moves, i.e, those of Table 6, take place with rates equal to p times those given in Table 6, whereas the P-related ‘left’ moves take place with rates equal to q times those given in Table 6, with $q = 1 - p$. For example:

$$\begin{aligned} AABC &\rightarrow ABAC \quad \text{with rate } pw_{AC}, \\ CBAA &\rightarrow CABA \quad \text{with rate } qw_{AC}. \end{aligned} \quad (3.12)$$

For this uniformly biased dynamics, the total entrance and exit rates for a given configuration read

$$\begin{aligned} W_{\text{in}}(\mathcal{C}) &= pW_{\text{in}}^{\text{right}}(\mathcal{C}) + qW_{\text{in}}^{\text{left}}(\mathcal{C}), \\ W_{\text{out}}(\mathcal{C}) &= pW_{\text{out}}^{\text{right}}(\mathcal{C}) + qW_{\text{out}}^{\text{left}}(\mathcal{C}), \end{aligned} \quad (3.13)$$

with self-explanatory notations. The stationarity condition for the totally asymmetric dynamics ($p = 1$) reads $W_{\text{in}}^{\text{right}}(\mathcal{C}) = W_{\text{out}}^{\text{right}}(\mathcal{C})$. On the other hand, P-related pairwise balance implies $W_{\text{in}}^{\text{right}}(\mathcal{C}) = W_{\text{out}}^{\text{left}}(\mathcal{C})$ and $W_{\text{out}}^{\text{right}}(\mathcal{C}) = W_{\text{in}}^{\text{left}}(\mathcal{C})$. We have therefore

$$W_{\text{in}}(\mathcal{C}) = W_{\text{in}}^{\text{right}}(\mathcal{C}) = W_{\text{in}}^{\text{left}}(\mathcal{C}) = W_{\text{out}}(\mathcal{C}) = W_{\text{out}}^{\text{right}}(\mathcal{C}) = W_{\text{out}}^{\text{left}}(\mathcal{C}). \quad (3.14)$$

These equations show that the partially asymmetric dynamics with uniform bias p has the same stationary-state measure as the totally asymmetric one. This dynamics interpolates between the symmetric (equilibrium) case ($p = 1/2$) and the totally asymmetric one ($p = 1$). The fact that the stationary-state measure is independent of the bias p thus appears as a general consequence of P-related pairwise balance.

4. Discussion

In this paper we explicitly constructed classes of nonequilibrium dynamics for two and three species of interacting particles, i.e., asymmetric stochastic dynamics which do not obey detailed balance, but whose nonequilibrium stationary-state measure is a prescribed measure. We have chosen to work with finite-temperature canonical Gibbs measures associated with spin Hamiltonians with nearest-neighbor interactions. The stationary current, as well as many other observables in the stationary state, can

therefore be evaluated, at least in principle, by means of the transfer-matrix formalism. We have emphasized the role of the various symmetries which can be imposed onto the dynamics.

For two species of interacting particles, a situation first considered by KLS [9], stationary-state measures are associated with the usual (anti)ferromagnetic Hamiltonian on the spin-1/2 Ising chain. Our result for the most general dynamics is given by Table 3. This dynamics has five free parameters, and does not obey pairwise balance in general. Only the cases considered by KLS, namely the totally asymmetric dynamics and the partially asymmetric dynamics with a uniform bias, obey P-related pairwise balance, where pairs of conjugate moves are related by parity.

We then turned to the novel situation of three species of interacting particles. Stationary-state measures are given by the most general Hamiltonian involving pairs of neighboring particles of the same species. This translates into a Blume-Emery-Griffiths spin-1 Hamiltonian. We first restricted the search of dynamics to the totally asymmetric case. The most general situation is described by Table 8. This dynamics obeys pairwise balance. It can therefore be extended to a partially asymmetric dynamics with uniform bias p . The three-parameter family of dynamics thus obtained interpolates between the symmetric (equilibrium) case ($p = 1/2$) and the totally asymmetric one ($p = 1$).

The most constrained class of stochastic dynamics we have investigated is the CP-invariant totally asymmetric one. For a prescribed stationary-state measure, there is indeed a uniquely defined such dynamics, with no free parameter, both for two species (see equation (2.29)) and for three species (see Table 7) of interacting particles.

Throughout this work we have put a strong emphasis on the numbers of free parameters in symmetric and asymmetric dynamics leading to a given stationary-state measure. Our results suggest the following rule: asymmetric stochastic dynamics leading to a given nonequilibrium stationary-state measure are far more constrained than symmetric dynamics leading to the same measure as an equilibrium measure.

To close up, let us demonstrate that the above empirical rule holds in a much broader class of stochastic models. To do so, we have chosen to put the results of this work in perspective with the following two characteristic examples, which also belong to the realm of driven diffusive systems.

Example 1.

Our first example is much in the spirit of the present paper. Consider a driven diffusive system consisting of K species of non-interacting particles on a ring, denoted by $I = A, B, \dots$, where $K \geq 2$ is arbitrary.

The most general exchange dynamics is defined by the $K(K - 1)$ rates u_{IJ} corresponding to the moves $IJ \rightarrow JI$ for $I \neq J$. We look for dynamics such that the stationary-state measure is uniform, i.e., all the configurations with given particle numbers N_I of each species are equally probable. This is indeed the right concept for a Gibbs measure in the absence of interactions, or, equivalently, in the limit of an infinite temperature.

The condition for having a uniform stationary-state measure reads

$$W_{\text{out}}(\mathcal{C}) - W_{\text{in}}(\mathcal{C}) = \sum_{IJ} u_{IJ}(N_{IJ} - N_{JI}) = 0 \quad (4.1)$$

for every configuration \mathcal{C} . The number of independent conditions on the rates imposed by this equation can be evaluated as follows. There are K^2 numbers of oriented pairs N_{IJ} , which obey the sum rules

$$\sum_J N_{IJ} = \sum_J N_{JI} = N_I, \quad \sum_I N_I = N. \quad (4.2)$$

Only $(K-1)^2$ pair numbers are therefore linearly independent. Equation (4.1) shown that each of the $(K-1)(K-2)/2$ antisymmetric combinations of these independent pair numbers yields one condition. The $K(K-1)$ rates therefore obey $(K-1)(K-2)/2$ conditions, so that the general asymmetric exchange dynamics for K species of non-interacting particles depends on

$$A_K = \frac{1}{2}(K-1)(K+2) \quad (4.3)$$

dimensionful parameters. On the other hand, for the symmetric exchange dynamics obeying the detailed balance property $u_{IJ} = u_{JI}$, the $K(K-1)/2$ rates are not constrained at all. Indeed (4.1) vanishes identically. The general symmetric exchange dynamics therefore depends on

$$S_K = \frac{1}{2}K(K-1) \quad (4.4)$$

parameters. One has

$$A_K = S_K + K - 1. \quad (4.5)$$

For two species ($K=2$), we have $A_2 = 2$ and $S_2 = 1$. There is no condition on the exchange rates, because there exists no antisymmetric combination of pair numbers. Equation (2.2) indeed implies $N_{AB} = N_{BA}$. As a consequence, the stationary-state measure is uniform for any value of the rates u_{AB} and u_{BA} . Interpreting A particles as particles and B particles as holes, we thus recover a known property of the ASEP [4, 5], namely that its stationary-state measure is uniform, irrespective of the bias.

For three species ($K=3$), we have $A_3 = 5$ and $S_3 = 3$. There is indeed one single antisymmetric combination of pair numbers:

$$Q = N_{AB} - N_{BA} = N_{BC} - N_{CB} = N_{CA} - N_{AC}. \quad (4.6)$$

There is accordingly a single condition on the six exchange rates for having a uniform stationary-state measure:

$$u_{AB} + u_{BC} + u_{CA} = u_{BA} + u_{CB} + u_{AC}. \quad (4.7)$$

This condition is known in the context of the matrix-product formalism [16]. It can be checked that (4.7) is fulfilled by the rates of Table 8 in the absence of interactions ($J_A = J_B = J_C = 0$). The only non-zero rates indeed read $u_{AB} = w_{IJ} = 1/2$, $u_{CB} = x_{IJ} = (1 + \alpha + \beta)/4$, and $u_{AC} = y_{IJ} = (1 - \alpha - \beta)/4$, irrespective of I, J . In the CP-invariant case, one has $u_{AB} = 1/2$ and $u_{CB} = u_{AC} = 1/4$.

Finally, for a large number of species ($K \gg 1$), asymmetric (driven, nonequilibrium) dynamics are far more constrained than symmetric (equilibrium) ones. Indeed the condition of having a uniform stationary-state measure roughly cuts off half the parameters, reducing their number from $K(K-1)$ to $A_K \approx K^2/2$, whereas for symmetric (equilibrium) dynamics the $K(K-1)/2$ rates are not constrained. The expression (4.5) shows that the difference $A_K - S_K \approx K \ll S_K$ is relatively negligible for a large number of species. In other words, for the uniform stationary-state measure, the full space of nonequilibrium dynamics is hardly larger than the subspace of equilibrium dynamics.

Example 2.

Our second example still belongs to the realm of driven diffusive systems, albeit with multiple occupancies. The results below strengthen our conclusion and broaden its range of applicability.

Consider the class of dynamical urn models defined as follows. N particles are distributed among M sites around a ring, with multiple occupancies. Let N_m be the number of particles at site $m = 1, \dots, M$. The system is subjected to the following stochastic dynamics.

- (i) a departure site d is chosen uniformly at random.
- (ii) a neighboring arrival site a is chosen as the right neighboring site ($a = d + 1$) with probability p , or the left neighboring site ($a = d - 1$) with probability $q = 1 - p$.
- (iii) a particle is transferred from site d to site a at a rate W_{kl} which only depends on the occupancies $k = N_d$ and $l = N_a$ of the two sites.

The relevant question in the present context is the following one. Under which conditions on the rates W_{kl} is the stationary-state measure a product measure of the form

$$P(\mathcal{C}) = P(N_1, \dots, N_M) = \frac{1}{Z_{M,N}} p_{N_1} \dots p_{N_M} \delta(N_1 + \dots + N_M, N)? \quad (4.8)$$

The answer to this question is known [17] (see [8] for a simple presentation). Consider first the case of an asymmetric dynamics ($p \neq 1/2$). The stationary-state measure is given by (4.8) if and only if the rates W_{kl} obey the two conditions

$$p_{k+1} p_l W_{k+1,l} = p_k p_{l+1} W_{l+1,k}, \quad (4.9)$$

$$W_{kl} - W_{k0} = W_{lk} - W_{l0}. \quad (4.10)$$

The first condition (4.9) relates the rates W_{kl} and the one-site factors p_k of the stationary-state measure distribution. The meaning of this relation is clear: it just expresses P-related pairwise balance. The second condition (4.10), which does not involve the p_k , is therefore more ‘kinematic’ than ‘dynamical’ in essence.

The zero-range process (ZRP) corresponds to the particular case where the rates $W_{kl} = u_k$ only depend on the occupation of the departure site. The condition (4.10) is then automatically satisfied, whereas (4.9) yields the following relation between the rates u_k and the factors p_k :

$$u_k = \omega \frac{p_{k-1}}{p_k} \quad (4.11)$$

for $k \geq 1$, where the constant ω fixes the time unit.

The most general dynamical urn model with stationary-state product measure is hardly more general than the ZRP. Let us state the following result, skipping the proof. For a given product measure of the form (4.8), with prescribed factors p_k , the general solution of (4.9) and (4.10) is entirely determined by the one-dimensional array of rates $\alpha_k = W_{k0}$. One has indeed (with $\alpha_0 = 0$)

$$W_{kl} = \frac{1}{p_k p_l} \sum_{m=0}^l p_{k+m} p_{l-m} (\alpha_{k+m} - \alpha_{l-m}). \quad (4.12)$$

The rates α_k are the rates at which an empty site ($l = 0$) is refilled, by receiving one particle from a non-empty neighboring site containing $k \geq 1$ particles.

In the case of a symmetric dynamics ($p = 1/2$), only the first condition (4.9) is requested [17, 8]. This relation expresses detailed balance. The resulting stationary state is therefore an equilibrium state. The condition (4.9) determines the rates W_{kl} for $k > l$ in terms of those for $k \leq l$.

For a general dynamical urn model, the stationary product measure thus depends on the one-dimensional array of rates α_k in the asymmetric case, and on the two-dimensional array of rates W_{kl} for $1 \leq k \leq l$ in the symmetric case.

To sum up, the two above examples of driven diffusive systems corroborate the picture which emerges from the results of the present work. Asymmetric (driven, nonequilibrium) stochastic dynamics producing a given stationary-state measure are far more constrained (in terms of numbers of free parameters) than symmetric dynamics producing the same measure as an equilibrium measure.

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